Identification of Geological Structures Using the Level Set Method

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ABSTRACT

We introduce a new inverse approach for efficiently identifying parameter structures (zonation) for the permeability field using the level set method, given spatially distributed observations of the permeability field (both lithology and/or permeability values) and hydraulic head measurements at various locations. In this preliminary study, the permeability field includes two different lithologies. In this method, the boundaries of zones are represented by a level set function. Starting from an initial choice, the boundary is then implicitly manipulated through the evolution of the level set function, which is sequentially optimized to match the observed data. No assumption has been made on the shape, size, locations, and the number of these zones, or the correlation structure and the proportion of different lithology. A synthetic example shows that this method can locate those embedded zones efficiently.

INTRODUCTION

Identifying parameter zonation is probably the most difficult in parameter identification problems. Traditionally, the heterogeneous domain of interest is divided into a number of zones and the parameter value in each zone is a constant to be determined. Although boundaries of these zones have significant impact on predicting flow and solute transport in the domain, in most cases we do not have enough direct information to infer the size, shape, locations, and the number of zones. *Sun and Yeh* [1985] were the first to propose a method to identify simultaneously both the parameter zonation and its parameter values for the hydraulic conductivity field. Using some model structure identification criteria, *Carrera and Neuman* [1986] were able to choose the best parameter zonation pattern among a number of given alternatives. *Eppstein and Dougherty* [1996] used a modified version of the extended Kalman filter to dynamically determine and refine zonation. *Tsai et al.* [2003] used Voronoi zonation to parameterize the unknown distributed parameter and solved the inverse problem by a sequential global-local optimization procedure. In this study, we introduce a new approach based on the level set method for zonation identification, applying the approach to a simple case of one material embedded in another. This method can be used to identify, for example, low-permeability layers in a relatively higher permeability porous media (or vice versa), or highly permeable fault zones in the subsurface.

The level set method is a powerful tool for solving problems that involve geometric evolution [*Osher and Sethian*, 1988]. The method has been used in several fields, including image segmentation [*Lie et al.*, 2005], shape optimization problems [*Burger*, 2003], and inverse problems [*Santosa*, 1996]. One of the advantages of the level set method is that it is much easier to work with a globally defined function than to keep track of the boundaries of regions of interest, which may split into many regions or merge into larger ones. It is important to emphasize that, comparing with geostatistical inverse methods such as indicator (co)kriging, the inverse approach based on the level set method requires no *a priori* assumptions on shape, size and locations of zones to be sought or correlation structures of these zones.

PROBLEM DESCRIPTION

We consider transient fluid flow in saturated porous media governed by the following equation

$$\nabla \cdot \left[K_s(\mathbf{x}) \nabla h(\mathbf{x},t) \right] + g(\mathbf{x},t) = S_s \frac{\partial h(\mathbf{x},t)}{\partial t}, \qquad (1)$$

subject to appropriate initial and boundary conditions. Here h(x,t) is the hydraulic head K_s is the hydraulic conductivity, g is source/sink term, S_s is the specific storage, $\mathbf{x} = (x_1, x_2, x_3)^T$ is the Cartesian coordinate, and t is the time. In this study, S_s is assumed to be a constant because its variation is relatively small comparing to that of the hydraulic conductivity. It is also assumed that the saturated hydraulic conductivity is a piecewise constant function, which is defined on a domain of interest that is partitioned into a number of unknown subdomains (zones). We do not have enough direct information regarding the exact size, shape, and locations of these zones. The problem is to identify parameter zonation, given observed lithologic types at some locations and head measurements at various locations and times.

LEVEL SET REPRESENTATION OF ZONES

Suppose that there is an unknown subset of Ω, denoted as D, representing the regions of low- (or high-) permeability zones. The boundary of D can be described by a function $\phi(\boldsymbol{x})$, $\partial D = \{\boldsymbol{x}: \phi(\boldsymbol{x}) = 0\}$. Accordingly, the hydraulic conductivity field can be represented by a binary function

$$K(\mathbf{x}) = \begin{cases} K_{\text{int}}, & \text{for } \mathbf{x} \in \mathrm{D} = \{\mathbf{x}: \phi(\mathbf{x}) < 0\} \\ K_{\text{ext}}, & \text{for } \mathbf{x} \in \mathrm{D}^{\mathrm{C}} = \{\mathbf{x}: \phi(\mathbf{x}) > 0\} \end{cases} \tag{2}$$

where $D^C = \Omega \backslash D$ is the complement of D, and K_{int} and K_{ext} are the hydraulic conductivities of interior and exterior of D, respectively. Note that for a given D there are an infinite number of functions that satisfy $\partial D = \{\boldsymbol{x}: \phi(\boldsymbol{x}) = 0\}$. However, any function $\phi(\boldsymbol{x})$ uniquely defines ∂D. This feature allows us to reinitiate $\phi(\boldsymbol{x})$ periodically without affect the boundary ∂D.

The inverse problem described above is to find $\phi(\boldsymbol{x})$ such that the head field solved using the spatial distribution of different zones characterized by $\partial D = \{\boldsymbol{x} : \phi(\boldsymbol{x}) = 0\}$ matches head measurements at various observation points and times. Note that no assumption has been made on connectivity of D, i.e., D could be a disjoint set, or connected but including a number of holes. In addition, there is no assumption made regarding the number of zones, their sizes and locations, correlation structures, or proportions of two materials.

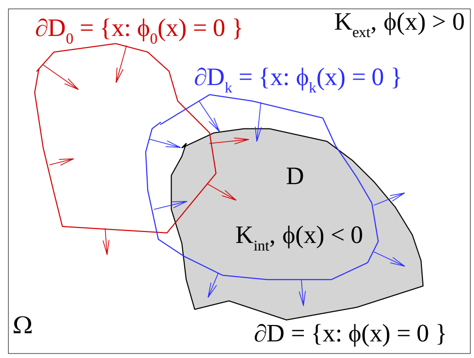

Fig. 1 Schematic diagram illustrating the evolution of boundaries.

Because the shape, size, and locations of set D are unknown, function $\phi(\boldsymbol{x})$ is also unknown. In the level set approach, we generate a sequence of functions $\phi_k(\boldsymbol{x})$, defining a sequence of regions $\partial D_k = \{\boldsymbol{x}: \phi_k(\boldsymbol{x}) = 0\}$, such that $D_k \rightarrow D$, as illustrated in Figure 1. In this iterative process, the number of zones and their shape, size, and locations are sequentially improved. The success of this method hinges on finding a strategy for efficiently propagating the boundary of D_k such that it approaches D.

FORMULATION OF INVERSE PROBLEM

The inverse problem described above can be written as a problem of minimizing an objective function:

$$F(K) = \frac{1}{2}\left\| H(K) - h_0 \right\| \quad , \tag{3}$$

where h_0 is a vector of head measurements, and $H(K)$ is an operator that maps the hydraulic conductivity field to a head field (by solving flow equations) and then takes head values at observation locations and times.

The directional derivative of function $F(K)$ in the direction δK, the variation of K, is given by

$$\delta F(K) = \int_D J^T(K)\left(H(K) - h_0\right)\delta K d\mathbf{x} \tag{4}$$

where $J(K)$ is the Jacobian of the head to hydraulic conductivity. In the discretized case, J is a $(n_h \times n_t) \times N$ Jacobian matrix, whose components are $J_{ij} = dh_i / dK_j$, $i = 1, 2, \ldots, n_h \times n_t$, $j = 1,2, \ldots, N$, where N is the number of grid nodes in the domain Ω.

For a given variation $\delta\phi$ (or $\delta\boldsymbol{x}$) that propagates D to a new set D', as shown in Figure 2, δK is non-zero only in $D'' = [D \cap D'^C] \cup [D' \cap D^C]$, which represents those points that are either in D but not in D' or in D' but not in D. As a result, the integral in (4) can be reduced to an integral over D''. For an infinitesimal $\delta\boldsymbol{x}$, $D'' = \partial D$, and (4) can be written as an integral over boundary ∂D. Certainly, variation δK is caused by variation $\delta\boldsymbol{x}$ (or $\delta\phi$), and we need to represent δK in terms of $\delta\boldsymbol{x}$ or $\delta\phi$. Following *Santosa* [1996], as illustrated in Figure 2, δK can be related to $\delta\boldsymbol{x}$ by $\delta K = (K_{int}-K_{ext})\, \boldsymbol{n}(\boldsymbol{x}) \bullet \delta\boldsymbol{x}\, |_{x \in \partial D}$, where $\boldsymbol{n}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x}) / |\nabla\phi(\boldsymbol{x})|$ is the normal direction of the curve at $\boldsymbol{x}$. We can assume that each point $\boldsymbol{x}$ on ∂D moves in the normal direction of boundary ∂D at this point, $\delta\boldsymbol{x} = \alpha(\boldsymbol{x})\, \boldsymbol{n}(\boldsymbol{x})$, where $\alpha(\boldsymbol{x})$ can be viewed as the propagation speed of the boundary at point $\boldsymbol{x}$. Substituting expressions of $\delta\boldsymbol{x}$ and δK into the integral

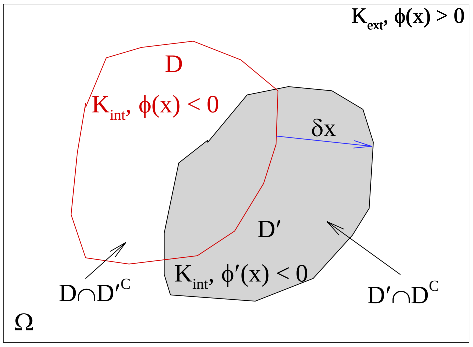

Fig. 2 Relationship between δK and δ**x**

$$\delta F(K) = \int_{\partial D} J^T(K)\big(H(K)-h_0\big)\big(K_{\text{int}}-K_{\text{ext}}\big)\alpha(\mathbf{x})d\mathbf{x} \tag{5}$$

To ensure that $\delta F(K)$ is negative during the iterative procedure, α is chosen as

$$\alpha(\mathbf{x}) = -\text{sgn}\big(K_{\text{int}}-K_{\text{ext}}\big)J^T\big(H(K)-h_0\big) \tag{6}$$

where *sgn* is the sign function, which returns the sign of its argument. Since the boundary is propagated at speed α, the direct measurements of the saturated hydraulic conductivity can be incorporated easily by setting the initial field to honor the measured hydraulic conductivity and then forcing the propagation speed to be zero at these measurement locations such that the boundary will not across these locations.

The physical meaning of (6) can be explained as follows. With no loss of generality, consider the case in which $K_{int} < K_{ext}$. Suppose that at point $\boldsymbol{x}$ at a given iteration, the modeled head values are too high compared to the measured values (i.e., $H(K) > h_0$), and that an increase of hydraulic conductivity value at $\boldsymbol{x}$ will increase the head at the observation points (i.e., $J(K) > 0$). In this case, the α value at point $\boldsymbol{x}$ computed from (6) will be positive, signifying that boundary ∂D at point $\boldsymbol{x}$ should move outwards. As a result, the hydraulic conductivity value at $\boldsymbol{x}$ decreases from K_{ext} to K_{int}, which will lead to a decrease in modeled head values in the new iteration.

Note that the propagation speed $\alpha(\boldsymbol{x})$ combines contributions from all observation points. If the sensitivity values of head at observation points $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ to hydraulic conductivity at $\boldsymbol{x}$ are the same, the point with larger head residual will exert more influence on the propagation speed in the next iteration. On the other hand, if the head residuals at two observation points (or times) are the same, the point with a higher sensitivity value will be more influential. Finally, should the modeled head match the observed head, the boundary would stop evolving.

The final step is to derive an equation for the evolution of $\phi(\boldsymbol{x})$. The variation of $\boldsymbol{x}$ can be related to that of ϕ by taking the variation of equation $\phi(\boldsymbol{x}) = 0$, $\delta\phi(\boldsymbol{x}) + \nabla\phi(\boldsymbol{x}) \bullet \delta\boldsymbol{x} = 0$. If the function $\phi(\boldsymbol{x})$ is expressed as a function of both $\boldsymbol{x}$ and an artificial time τ, $\phi = \phi(\boldsymbol{x}, \tau)$, the evolution of $\phi(\boldsymbol{x}, \tau)$ accordingly defines the evolution of boundary $\partial D(\tau) = \{\boldsymbol{x} : \phi(\boldsymbol{x}, \tau) = 0\}$. For sufficiently large τ, $\partial D = \{\boldsymbol{x} : \phi(\boldsymbol{x}, \tau) = 0\}$ defines the spatial distribution of two materials and gives the solution to the inverse problem. Substituting expressions of $\delta\boldsymbol{x}$ into above variational relationship and recalling the definition of $\boldsymbol{n}(\boldsymbol{x})$, an initial value problem for $\phi(\boldsymbol{x}, \tau)$ is derived,

$$\frac{\partial \phi(\mathbf{x},\tau)}{\partial \tau} + \alpha(\mathbf{x},\tau)\left|\phi(\mathbf{x},\tau)\right| = 0, \quad \phi(\mathbf{x},0) = \phi_0(\mathbf{x}) \tag{7}$$

which is called the level set equation. The solution of this equation over the artificial time τ gives the boundary $\partial D(\tau)$ and eventually the solution stablizes, representing the solution to the inverse problem.

ILLUSTRATIVE EXAMPLES

In this section, we illustrate the level set method with a two-dimensional synthetic example. The test system consists of steady-state, saturated water flow in a rectangular domain of 100 m × 100 m, discretized into elements of a size 1 m × 1 m. The true (synthetic) permeability field is shown in Figure 3(a), in which a lower-permeable layer (k = 10^{-13} m^2) is embedded in the background material (k = 10^{-10} m^2), offset by a fault, and observed at two boreholes. Lacking other constraining information, the geometry of the lower-permeable unit based on the traditional geological correlation will be incorrectly interpreted as a laterally connecting bed in the region between the two dashed lines. No matter how many observed head or concentration data are available, the parameter estimates from model calibration using this incorrect geological model certainly will yield erroneous results, and thus the prediction of flow and solute transport based on these parameter estimates will also be incorrect. The boundary conditions are prescribed as constant head at left (H_1 = 10.5 m) and right (H_2 = 10.0 m) boundaries and no flow at the two lateral boundaries. The steady-state flow equation is solved for the synthetic permeability field, and 36 steady-state head measurements at various locations (see Fig. 3a) are assumed to be available.

Since the lower-permeable unit has been observed at two wells, we initialize the iterative procedure by choosing an initial setting of lower-permeable zones as depicted in Fig. 3(b) and the initial level set function $\phi_0(\boldsymbol{x})$ is choosing as a signed distance function with this initial boundary of lower-permeable unit. The flow equation is then solved using the initial permeability field and the pseudo velocity field $\alpha(\boldsymbol{x})$ is computed based on the difference between the modeled head and the observed head using (6). Finally, the boundary is propagated by solving the level set equation (7). The new boundary of low-permeability zones is determined by the set $\partial D = \{\boldsymbol{x} : \phi(\boldsymbol{x}) = 0\}$. This process is repeated until either the prescribed number of updates has been reached or the difference between the modeled head and the observed head is smaller than a prescribed value. To prevent steep or flat gradients of the level set function $\phi(\boldsymbol{x})$, the function is periodically reinitialized to a signed distance function, i.e., $\phi(\boldsymbol{x}) = -d(\partial D, \boldsymbol{x})$ for $\boldsymbol{x} \in D$ or

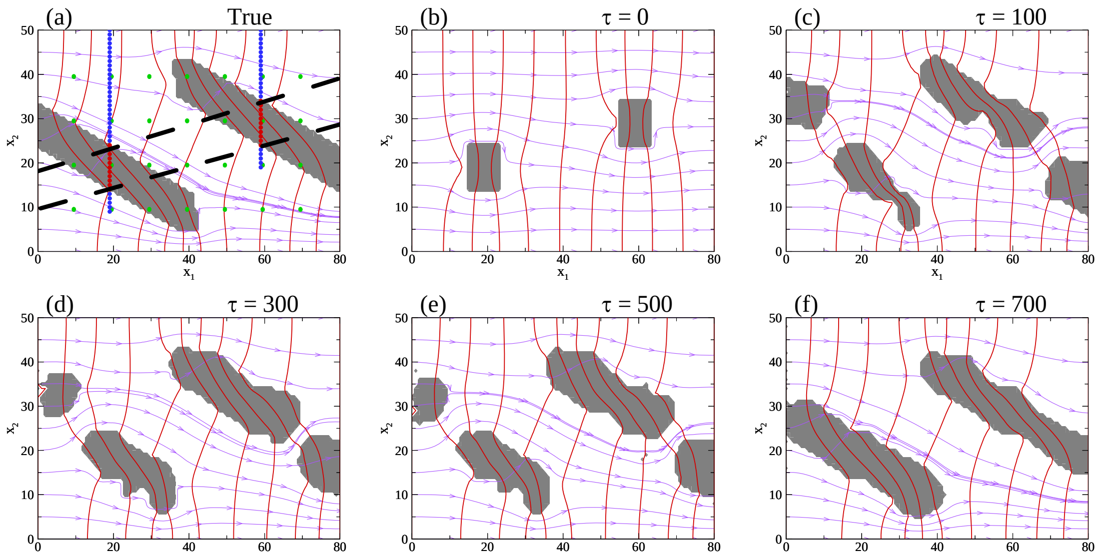

Figure 3. An illustrative example. (a) The problem configuration. The region between two dashed-lines is the possible lower-permeable zone identified by the traditional geological correlation. (b) Initial lower permeable zones in the proposed approach, based on borehole observation. (c-f) Identified lower-permeable units at several iteration steps.

$\phi(\boldsymbol{x}) = d(\partial D, \boldsymbol{x})$ for $\boldsymbol{x} \in D^C$, where d represents the distance between point $\boldsymbol{x}$ and boundary ∂D. Note that this reinitialization will not change the boundary ∂D. The evolution of the boundary ∂D is depicted in Figure 3c-3f, where the artificial time τ represents the number of updates. The final inversion results (Fig. 3f) are very similar to the true structures and head distribution, indicating that the level set method can be used to efficiently identify parameter zonation.

SUMMARY AND FURTHER WORK

This paper is a preliminary study proposing a new inverse algorithm based on the level set method. In this method, the boundary of low- (or high-) permeability zones is represented by the zero level function. Starting from an initial setting of zones, their boundary is sequentially evolved to reduce the difference between the observed hydraulic head and the modeled head. The propagation speed of the boundary at any iteration is proportional to the sensitivity of head to the permeability field and the residual between the observed and modeled head at various measurement locations and observation times. The synthetic example presented showed that the level set method can efficiently identify the parameter structure.

These promising initial results suggest that further work is warranted to explore the level set method in more detail. Future work will include extending the method to incorporate transient head response data; using other data sets beyond simply head data (solute concentration or travel time measurements); consideration of measurement error and data density into the evaluation of the method; and developing methods for the joint inversion for shape and permeability of the individual zones. Furthermore, for maximum usefulness, the method should be extended to account for an arbitrary number of stratigraphic units, each with a distinct unknown value of permeability. Finally, one important issue that needs to be addressed in the future is the uncertainty associated with the prediction of internal boundaries between different materials, which may provide direct input to the random domain decomposition method of Winter and Tartakovsky [2000] for better quantifying flow and solute transport in subsurface..

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